Non-$\mathbb{Q}$-factorial Singularities

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The location and the characters

Algebraic geometry The garden where our quest takes place.

Birational geometry The comprehensive parent.

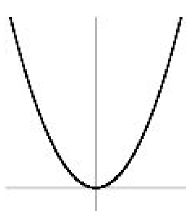
Minimal Model Program The discerning parent.

Singularity theory The main character, the ill-behaved daughter.

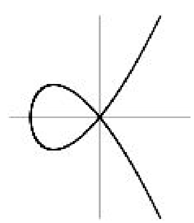
Divisors The ambivalent servants.

Prologue

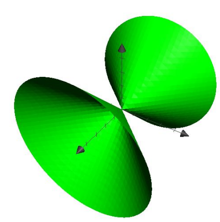
Algebraic Geometry is the study of the geometry of the zeros of polynomial equations, using algebraic tools. Some examples of polynomial equations are the line the parabola $y = x^2$, the nodal curve $y^2 = x^2(x + 1)$, or the cone $z^2 = x^2 - y^2$. These objects are called *algebraic varieties*.



$y = x^2$



$y^2 = x^2(x + 1)$



$z^2 = x^2 - y^2$

One of the first questions we can ask ourselves is how to classify algebraic varieties. There are always different sides to this question: how do we group varieties in classes, how do we choose a representative in each class, and finally how do different representatives relate to each other. In *birational geometry* each class is given by varieties that have isomorphic open dense subsets (i.e. varieties that are *birational*). For example the parabola and the nodal curve (the first two examples above) belong to the same class. The *Minimal Model Program* prescribes which properties a representative in each birational class should have and it is essentially an algorithmic way of choosing such representative, which is called a *minimal model*. Finally, to answer the third part of the classification problem, techniques of *moduli theory* or *deformation theory* can be used.

It is when working with the minimal model (the good representative) that *singularity theory* will naturally arise. Indeed, the minimal model is usually singular. The last two examples we gave above, the nodal curve and the cone, are typical examples of *singular varieties*. A standard way of studying singular varieties is using divisors. A *divisor* is a "codimension 1" subvariety. However, there are two possible meanings for "codimension 1". Geometrically, it could mean that we consider a variety of dimension 1 less. In the cone $z^2 = x^2 - y^2$, a line through the origin is a variety of codimension 1. This is an example of a *Weil divisor*. Algebraically, "codimension 1" means that the variety is (locally) given by just one equation. These are called *Cartier divisors*. If a subvariety is described by just one equation, it will also have geometric codimension 1; this means that all Cartier divisors determine Weil divisors. However, the converse is not true. For example, the line through the origin $z = x - y = 0$ in the aforementioned cone cannot be describe by just one equation: this is an example of a Weil divisor which is not Cartier (notice that twice the line can be described by just one equation). The singularity theory I am interested is precisely when these two types of divisors are very different; such singularities are called *non-$\mathbb{Q}$-factorial* and are vey badly behaved singularities. These singularities appear in the Minimal Model Program, even if the minimal model itself does not have them, much like you need complex numbers to give a formula for the roots of a cubic real polynomial, even when all the roots are real.