Vector fields and Forms

Contents

These are notes on vector fields and forms. They are intended as a complement for the book, not as a substitute. You still need to look at the chapter of the book dedicated to vector fields.

1 Vector fields and forms

Historically, the theory of the integration in several variables used the language of *vector fields*; nowadays, we prefer to talk about *forms*. Forms were modelled on vector fields, and so were operations on them, but they are a more natural object when you want to integrate.

A two-dimensional ***vector field*** on a region D in the plane $\mathbb{R}^2$ is a "function" that to each point of D associates a vector of dimension 2, or, if you prefer, is a pair of functions on D. A two dimensional vector field is of the form

$$\vec{F} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j} = \langle P(x,y), Q(x,y)\rangle \tag{1.1}$$

(where P and Q are functions on D).

A three-dimensional ***vector field*** on a region D in the space $\mathbb{R}^3$ is a "function" that to each point of D associates a vector of dimension 3, or, if you prefer, is a collection of three functions on D. A three dimensional vector field is of the form

$$\vec{F} = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle \tag{1.2}$$

(where P, Q and R are functions on D).

Above we gave the definition of vector fields in dimension 2 and 3, but the same definitions extend to any dimension.

Example 1.1. Jim is running at the constant speed of 10 miles per hour, from his place to campus. If we denote his path by a curve C, then for each point of the curve C we have a velocity $\vec{v}(x,y)$ (the velocity of Jim). The velocity of Jim is an example of vector field on C. We observe that the fact that Jim has a constant speed of 10 miles per hour is equivalent to say that, if $\vec{v} = P\,\vec{i} + Q\,\vec{j}$, then $|\vec{v}| = \sqrt{P^2+Q^2} = 10$. This is called the ***magnitude*** of the vector field.

Example 1.2. In the previous example we saw a vector field with constant magnitude. This is not the general case. If we consider a rock falling (with no constrains), then the velocity of the rock is a vector field as well. However, since the motion is accelerated, the speed is non constant. And the speed is precisely the magnitude of the velocity.

Exercise 1. *Let $\vec{r}\,(t) = \langle t, t^2 - e^t, 5\rangle$ be the position of a particle as a function of time t. Find the velocity of the particle and write it as a three dimensional vector field.*

Now we will introduce the notion of form. The definition will be very imprecise.

A ***form*** is "everything where a d appears, followed by a variable". In particular, we give the following two definitions:

A two dimensional 1-***form*** on a region D in the xy-plane is something that looks like

$$f(x,y)\,dx + g(x,y)\,dy, \tag{1.3}$$

where f and g are functions on D.

A three dimensional 1-***form*** on a region D in the xyz-space (the space $\mathbb{R}^3$) is something that looks like

$$f(x, y, z)\,dx + g(x, y, z)\,dy + h(x, y, z)\,dz \tag{1.4}$$

where f, g and h are functions on D.

Example 1.3. An example of two dimensional 1-form is $2\,dx + 4\,dy$. Another example is $(3x + y)\,dx + \sqrt{x^2 + y^2}\,dy$. Also $y\,dx$ is a 1-form.

Example 1.4. An example of three dimensional 1-form is $z\,dx + x\,dy + y\,dz$.

Example 1.5. In a form any variable can appear. If in our problem or context, t is a variable, dt is a 1-form, and so might be $dx + dt$.

Notice that you do not sum the functions f, g and h together, but the dx, dy and dz keep them separate, exactly as for vector fields. In this sense, we can think of dx as a different name for $\vec{i}$ ($\vec{i}$ is in the x-direction), dy as a different name for $\vec{j}$ ($\vec{j}$ is in the y-direction) and dz as a different name for $\vec{k}$ ($\vec{k}$ is in the z-direction). Then the vector field $P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$ corresponds to the 1-form $P\,dx + Q\,dy + R\,dz$, and vicersa. The same is true for two dimensional vector fields and two dimensional 1-forms. We have

$$P\,\vec{i} + Q\,\vec{j} \leftrightarrow P\,dx + Q\,dy \tag{1.5}$$

and

$$P\,\vec{i} + Q\,\vec{j} + R\,\vec{k} \leftrightarrow P\,dx + Q\,dy + R\,dz. \tag{1.6}$$

This natural idea is actually a theorem, and is called the ***musical isomorphism***.

Exercise 2. *Write the* 1*-form corresponding to the vector field*

$$\sqrt{x + y}\,\vec{i} - y\,\vec{j}.$$

What is its dimension (two or three)?

Exercise 3. *Write the vector field corresponding to the* 1*-form*

$$3\,dx + 5\,dy + x\,dz,$$

and compute its magnitude.

2 Integration of vector fields and 1-forms

Following the correspondence of the previous section, we define the integration of vector fields. In general, if we have a vector field $\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$, corresponding to the form $P\,dx + Q\,dy + R\,dz$, on a region D, and C is a curve in D, we define the integral of $\vec{F}$ on C as

$$\int_C \vec{F} \cdot d\vec{s} = \int_C P\,dx + Q\,dy + R\,dz. \tag{2.7}$$

To justify the notation above, you can think of the vector $d\vec{s}$ as the vector representing the variation in the coordinates, that is, the formal vector

$$d\vec{s} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k} = \langle\, dx,\, dy,\, dz\rangle. \tag{2.8}$$

Then

$$\vec{F} \cdot d\vec{s} = P\,dx + Q\,dy + R\,dz. \tag{2.9}$$

This gives us a new interpretation of the musical isomorphism, that is that the 1-form corresponding to a vector field is the (formal) dot product of $\vec{F}$ and $d\vec{s}$.

To practically compute the integral of a vector field, we need a parametrization of the curve C. Let us say that the curve C is parametrized by $\vec{r}(t) = \langle l(t), m(t), n(t)\rangle$, with $a \le t \le b$. Then we compute

$$\int_C P\,dx + Q\,dy + R\,dz = \\ = \int_a^b \Big[P(l(t))l'(t) + Q(m(t))m'(t) + R(n(t))n'(t)\Big]dt. \tag{2.10}$$

The idea here is that, if C is parametrized by $\vec{r}(t)$, then x is a function of t, namely $x = l(t)$. Therefore

$$dx = \frac{dx}{dt}\,dt = \frac{dl}{dt}\,dt = l'(t)\,dt;$$

and similarly for dy and dz. We can say that, since we are on the curve C, $\vec{s} = \vec{r}$ and

$$d\vec{s} = d\vec{r} = \langle l'(t), m'(t), n'(t)\rangle\,dt = \vec{r}\,'(t)\,dt \tag{2.11}$$

and therefore we have the more compact notation

$$\int_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt. \tag{2.12}$$

For two dimensional vector fields in the plane, it is exaclty the same thing (think of $R = 0$ and $dz = 0$).

Example 2.1. Let $\vec{F}$ be the vector field $\vec{F} = x^2\,\vec{i} + y^2\,\vec{j}$ on the whole plane, and let C be a circle of radius R centered at the origin (and considered only once counterclockwise). We want

$$\int_C \vec{F} \cdot d\vec{s}.$$

The parametrization of C is $\vec{r}(t) = \langle R\cos t, R\sin t\rangle$ with $0 \le t \le 2\pi$. Then

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{s} &= \int_C x^2\,dx + y^2\,dy = \\ &= \int_0^{2\pi} R^2\cos^2 t(-R\sin t) + R^2\sin^2 t(R\cos t)\,dt = \\ &= R^3 \int_0^{2\pi} -\sin t\cos^2 t + \cos t\sin^2 t\,dt.\end{aligned}$$

We write this integral as

$$\begin{aligned}&R^3 \int_0^{2\pi} -\sin t\cos^2 t + \cos t\sin^2 t\,dt = \\ &\qquad = R^3 \int_0^{2\pi} -\sin t\cos^2 t\,dt + R^3 \int_0^{2\pi} \cos t\sin^2 t\,dt,\end{aligned}$$

and we compute the two integrals separately. For the first one, we split it in

$$R^3 \int_0^{2\pi} -\sin t\cos^2 t\,dt = R^3 \int_0^{\pi} -\sin t\cos^2 t\,dt + R^3 \int_{\pi}^{2\pi} -\sin t\cos^2 t\,dt.$$

If we use the substitution $u = \cos t$, $du = -\sin t\,dt$, the first part of the first integral becomes

$$R^3 \int_0^{\pi} -\sin t\cos^2 t\,dt = R^3 \int_1^{-1} u^2 du = R^3 \left[\frac{u^3}{3}\right]_1^{-1} = -\frac{2}{3}R^3.$$

With the same substitution, the second part of the first integral gives

$$R^3 \int_{\pi}^{2\pi} -\sin t\cos^2 t\,dt = R^3 \int_{-1}^{1} u^2 du = \frac{2}{3}R^3.$$

Thus the first integral is zero. Similarly, we split the second integral in

$$\begin{aligned}&R^3 \int_0^{2\pi} \cos t\sin^2 t\,dt = \\ &= R^3 \int_0^{\pi/2} \cos t\sin^2 t\,dt + R^3 \int_{\pi/2}^{3\pi/2} \cos t\sin^2 t\,dt + R^3 \int_{3\pi/2}^{2\pi} \cos t\sin^2 t\,dt.\end{aligned}$$

With the substitution $u = \sin t$, we find that also this integral is zero. So we found in our case

$$\int_C \vec{F} \cdot d\vec{s} = 0.$$

Exercise 4. *Let $\vec{F}$ be the vector field $\vec{F} = -y\,\vec{i} + x\,\vec{j}$, and let C be the circle centered at the origin of radius R and considered counterclockwise only once. Find*

$$\int_C \vec{F} \cdot d\vec{s}.$$

Exercise 5. *Let $\vec{F}$ be the vector field $\vec{F} = -y\,\vec{i} + x\,\vec{j}$, and let C be the ellipse parametrized by $\vec{r}\,(t) = \langle 2\cos t, 5\sin t\rangle$, with $0 \leq t \leq 2\pi$. Find*

$$\int_C \vec{F} \cdot d\vec{s}.$$

Exercise 6. *Let $\vec{F}$ be the vector field $\vec{F} = -y\,\vec{i} + x\,\vec{j} + z\,\vec{k}$, and let C be the arc of helix parametrized by $\vec{r}\,(t) = \langle \cos t, \sin t, t\rangle$, with $0 \leq t \leq 2\pi$. Find*

$$\int_C \vec{F} \cdot d\vec{s}.$$

3 Operations with forms

Here we start a more abstract part of these notes, but do not fear, it is actually not too hard.

For the sake of simplicity in the notations, we use the following definition:

A 0-***form*** on a region D in the xy-plane is a function $f(x, y)$ defined on D.

A 0-***form*** on a region D in the xyz-space (that is, $\mathbb{R}^3$) is a function $f(x, y, z)$ defined on D.

We have three basic operations that we can do with forms: adding, multiplying, differentiate. All of them are functional to the integration. We will discuss the first two in this section, and the last one in the next.

3.1 Sum

To ***add*** two 1-forms is simple: if $f\,dx$ is a 1-form and $g\,dy$ is another, then $f\,dx + g\,dy$ is their sum. In this sense, the expected properties of factorizations apply: $f\,dx + g\,dx = (f + g)\,dx$ and $f\,dx + f\,dy = f\,(\,dx + \,dy)$ (it is f times $dx + \,dy$, not f applied to $\,dx + \,dy$).

Example 3.1. The sum of $3\,dx + \,dy$ and $x\,dx + z\,dz$ is

$$(3\,dx + \,dy) + (x\,dx + z\,dz) = (3 + x)\,dx + \,dy + z\,dz.$$

Exercise 7. *Compute the sum*

$$(\cos y\,dx) + (\sin y\,dy + \,dz) - (\cos y\,dx + \sin y\,dy).$$

On the other hand, we already know how to add to functions, and so we know how to add two 0-forms. Later in this section we will define 2-forms and 3-forms, and the sum between those will be natural as well. We point out that ***you can only add forms if they are of the same type***. Explicitely, you cannot add a 0-form and a 1-form, or a 1-form and a 2-form, ... The reason is that when we work with forms we are thinking of integrating. Let say that we have the 0-form 3 and the 1-form dx. If their sum existed, it should be $3 + dx$. But then $\int_0^1 3 + dx$ does not make sense, since we cannot integrate $\int_0^1 3$ (we need a dx).

3.2 Wedge product

The ***multiplication*** is more complicated. Since forms were modeled on vector fields, the idea of multiplying two forms is to generalize the cross product. The problem of cross product is that it exists only in dimension 3, while we are interested in working in any dimension. In order to do this, we introduce the ***wedge product***. If $\vec{a}$ and $\vec{b}$ are vectors, then we know that $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$ (the cross product is antisymmetric) and that $\vec{a} \times \vec{a} = 0$. If now u and v are any two variables and du and dv are 1-forms (think for example of $u = x$, y or z and $v = x$, y, or z), then we define

$$dv \wedge du = -du \wedge dv \tag{3.13}$$

and

$$du \wedge du = 0. \tag{3.14}$$

Also, if f is a 0-form (a function) we simply say that

$$f \wedge du = f du. \tag{3.15}$$

The usual properties of distributivity and associativity apply. We can use these rules to compute the product of more then two forms, and move them around.

Example 3.2. For example,

$$\begin{aligned} dx \wedge dy \wedge dx &= (dx \wedge dy) \wedge dx = (-dy \wedge dx) \wedge dx = -dy \wedge dx \wedge dx = \\ &= -dy \wedge (dx \wedge dx) = -dy \wedge 0 = 0. \end{aligned}$$

This means that every time that we have a wedge product and at least two are equal, the product is zero.

Exercise 8. *Using the properties of the wedge product, show that*

(a) $dx \wedge dy \wedge dz = dy \wedge dz \wedge dx = dz \wedge dx \wedge dy$;

(b) $dx \wedge dz \wedge dy = dz \wedge dy \wedge dx = dy \wedge dx \wedge dz$;

(c) $dx \wedge dy \wedge dz = -dy \wedge dx \wedge dz$.

Now that we have the wedge product we can define 2-forms and 3-forms.

A ***2-form*** on a region D in the xy-plane is something that looks like

$$f(x,y)\,dx\wedge\,dy, \tag{3.16}$$

where $f(x,y)$ is a function defined on D.

A ***2-form*** on a region D in the xyz-space is something that looks like

$$f(x,y,z)\,dx\wedge\,dy+g(x,y,z)\,dy\wedge\,dz+h(x,y,z)\,dz\wedge\,dx, \tag{3.17}$$

where f, g, and h are functions defined on D.

A ***3-form*** on a region D in the xyz-space is something that looks like

$$f(x,y,z)\,dx\wedge\,dy\wedge\,dz, \tag{3.18}$$

where f is a function defined on D.

As in the case of 1-forms, x, y and z need not to be the only variables. For example, if we are working in polar coordinates ρ,θ, the product $d\rho\wedge\,d\theta$ is a 2-form.

As 1-form are what we integrate in simple integrals, 2-forms are the natural object when we do double integrals and 3-form are the one for triple integrals. You can think that when you write the double integral

$$\int\!\!\int_D f\,dx\,dy$$

you are actually thinking of

$$\int\!\!\int_D f\,dx\wedge\,dy$$

and omitting the wedge symbol. The same is true for triple integrals. In this interpretation, $dx\wedge\,dy$ represents the element of area of the "rectangle" given by a dx and a dy. Therefore we have another reason why $dx\wedge\,dx=0$ (indeed, if you consider just dx you do not have any "rectangle" and therefore you do not have any area).

Before proceeding, we need to observe two things. The first one is that the analogy between the wedge product and the cross product is good only for defining the properties of the former. Indeed, in the space we have $\vec{i}\times\vec{j}=\vec{k}$ while $dx\wedge\,dy\neq\,dz$ (the first one is a 2-form, the second one is a 1-form). The second observation is that the definition of 2-forms and 3-forms uses implicitely some of the properties of the wedge product. For example, in the definition of 2-form the product $dy\wedge\,dx$ does not appear because

$$f\,dx\wedge\,dy+g\,dy\wedge\,dx=f\,dx\wedge\,dy+g(-\,dx\wedge\,dy)=(f-g)\,dx\wedge\,dy$$

and we do not need it. Also $dx \wedge dx$ and $dy \wedge dy$ do not appear since they are both 0. Here we used the sum between 2-forms. As mentioned before, this operation is natural, and so is the one between 3-forms.

The multiplication between two generic forms uses the $\wedge$ product (and its properties). This can be illustrated in an example.

Example 3.3. Let $\omega = 3\,dx + 5\,dy$, $\eta = x\,dx + \sin x\,dy$ and $\theta = z^2\,dz \wedge dx$. Then

$$\begin{aligned}\omega \wedge \eta &= (3\,dx + 5\,dy) \wedge (x\,dx + \sin x\,dy) = \\ &= 3x\,dx \wedge dx + 3\sin x\,dx \wedge dy + 5x\,dy \wedge dx + 5\sin x\,dy \wedge dy.\end{aligned}$$

Now, $dx \wedge dx = dy \wedge dy = 0$ and $dy \wedge dx = -\,dx \wedge dy$, so we can rewrite the above product as

$$\begin{aligned}\omega \wedge \eta &= 3x\,dx \wedge dx + 3\sin x\,dx \wedge dy + 5x\,dy \wedge dx + 5\sin x\,dy \wedge dy = \\ &= 3\sin x\,dx \wedge dy + 5x\,dy \wedge dx = \\ &= 3\sin x\,dx \wedge dy - 5x\,dx \wedge dy = \\ &= (3\sin x - 5x)\,dx \wedge dy.\end{aligned}$$

Remember to keep track of the order of dx and dy in the multiplication, it is very important!

We can also compute,

$$\omega \wedge \theta = (3\,dx + 5\,dy) \wedge (z^2\,dz \wedge dx) = 3z^2\,dx \wedge dz \wedge dx + 5z^2\,dy \wedge dz \wedge dx.$$

Since $dx \wedge dz \wedge dx$ has two parts that are equals, $dx \wedge dz \wedge dx = 0$. Also, using exercise 8, $dy \wedge dz \wedge dx = dx \wedge dy \wedge dz$, so that

$$\begin{aligned}\omega \wedge \theta &= 3z^2\,dx \wedge dz \wedge dx + 5z^2\,dy \wedge dz \wedge dx = \\ &= 5z^2\,dy \wedge dz \wedge dx = \\ &= 5z^2\,dx \wedge dy \wedge dz.\end{aligned}$$

Two very important properties of the wedge product are that if ω and η are ***any*** forms, then

$$\eta \wedge \omega = -\omega \wedge \eta \tag{3.19}$$

and

$$\omega \wedge \omega = 0. \tag{3.20}$$

We will illustrate that in the next example on two special cases.

Example 3.4. As before, let $\omega = 3\,dx + 5\,dy$ and $\eta = x\,dx + \sin x\,dy$. Then

$$\omega \wedge \eta = (3\sin x - 5x)\,dx \wedge dy.$$

With the same computation as before,

$$\eta \wedge \omega = (5x - 3\sin x)\, dx \wedge\, dy = -\omega \wedge \eta.$$

If we compute $\omega \wedge \omega$ we find

$$\begin{aligned}\omega \wedge \omega &= (3\, dx + 5\, dy) \wedge (3\, dx + 5\, dy) = \\ &= 9\, dx \wedge\, dx + 15\, dx \wedge\, dy + 15\, dy \wedge\, dx + 25\, dy \wedge\, dy.\end{aligned}$$

Now, $dx \wedge\, dx = \, dy \wedge\, dy = 0$ and $dy \wedge\, dx = -\, dx \wedge\, dy$. Thus

$$\begin{aligned}\omega \wedge \omega &= 9\, dx \wedge\, dx + 15\, dx \wedge\, dy + 15\, dy \wedge\, dx + 25\, dy \wedge\, dy = \\ &= 15\, dx \wedge\, dy + 15\, dy \wedge\, dx = \\ &= 15\, dx \wedge\, dy - 15\, dx \wedge\, dy = \\ &= 0.\end{aligned}$$

Exercise 9. *Compute the following products (simplify as much as possible):*

(a) $(\sin x\, dx + z\, dy - y^2\, dz) \wedge (z\, dx - \, dy)$*;*

(b) $(3\, dx + 5\, dy - 2\, dz) \wedge (5\, dx + 2\, dy + 3\, dz) \wedge (5\, dy)$*;*

(c) $(3\, dx \wedge\, dy + 2\, dy \wedge\, dz) \wedge (\sin x\, dx + \sin y\, dy + \sin z\, dz)$*;*

(d) $(\, dz \wedge\, dx + 2\, dx \wedge\, dy) \wedge (2\, dy \wedge\, dz)$*;*

(e) $(3\, dx + 5\, dy - 2\, dz) \wedge (5\, dx + 2\, dy + 3\, dz) \wedge (5\, dy) \wedge (2\, dx + 3\, dy - 5\, dz)$*.*

Exercise 10. *Using the properties discussed above, if ω and η are any two forms of the same type, compute*

$$(\omega - \eta) \wedge (\omega + \eta)$$

(notice the difference from the usual product $(a - b)(a + b)$).

4 Differentiation of forms

The other operation is to ***differentiate***. If $f(x)$ is a function, then you can think of it as a substitution $u = f(x)$ in an integral, and then you have that $du = f'(x)\, dx$. Thinking of the line integrals of vector fields (or 1-form), we also find $dx = l'(t)\, dt$. The idea behind the differentiation of forms is to generalize this process.

If $f(x)$ is a 0-form, we define $df = \frac{\partial f}{\partial x}\, dx$; if $f(x, y)$ is a 0-form, then $df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy$; if $f(x, y, z)$ is a 0-form, then

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz \tag{4.21}$$

(the first two cases are particular one of this last one). In general, we consider the partial derivatives of the function with respect to all possible variables, and we sum them all up, each of them with d and the corresponding variable.

Example 4.1. If $f = 3x - y^2$, then

$$df = 3\,dx - 2y\,dy.$$

Example 4.2. Just to be precise, again we can have variables other than x, y of z. If $f = 3x - t + \sin w$, then

$$df = 3\,dx - \ dt + \cos w\,dw.$$

Example 4.3. If $f = \sin(xy) - xz^2$, then

$$df = (y\cos(xy) - z^2)\,dx + x\cos(xy)\,dy - 2xz\,dz.$$

Exercise 11. *For the following examples of functions, compute their differential:*

(a) $f = 3x + 2y$;

(b) $f = e^{xy} - z^2$;

(c) $f = \cos(x/y) - xz + y^2$.

We saw that any form has a basic term a function followed by wedge products of d of variables. Then, we define the differential of that term as d of the function, multiplied with wedge product with what is left. For 1-forms this becomes, if

$$\omega = P\,dx + Q\,dy + R\,dz,$$

then

$$d\omega = dP \wedge\, dx + dQ \wedge\, dy + dR \wedge\, dz. \tag{4.22}$$

Example 4.4. For example, let us consider the form

$$\omega = xy\,dx = f\,dx,$$

where $f = xy$. Then

$$\begin{aligned} d\omega = df \wedge\, dx &= \left(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy\right) \wedge\, dx = \\ &= (y\,dx + x\,dy) \wedge\, dx = y\,dx \wedge\, dx + x\,dy \wedge\, dx = \\ &= -x\,dx \wedge\, dy. \end{aligned}$$

Example 4.5. If $\omega = x^2y\,dx - 2y\,dz$, then

$$\begin{aligned} d\omega &= d(x^2y) \wedge dx - d(2y) \wedge dz \\ &= (2xy\,dx + x^2\,dy) \wedge dx + (2\,dy) \wedge dz = \\ &= 2xy\,dx \wedge dx + x^2\,dy \wedge dx + 2\,dy \wedge dz = \\ &= -x^2\,dx \wedge dy + 2\,dy \wedge dz. \end{aligned}$$

Example 4.6. If $\omega = (x + y^2 + z)\,dx + (y + x^2)\,dz$, then

$$\begin{aligned} d\omega &= (\,dx + 2y\,dy + \,dz) \wedge dx + (\,dy + 2x\,dx) \wedge dz = \\ &= dx \wedge dx + 2y\,dy \wedge dx + \,dz \wedge dx + \,dy \wedge dz + 2x\,dx \wedge dz = \\ &= -2y\,dx \wedge dy + \,dy \wedge dz + (1 - 2x)\,dz \wedge dx. \end{aligned}$$

Exercise 12. *For each of the following 1-forms, compute the differential*

(a) $\omega = (x + y^2)\,dx + (y + z^2)\,dy + (z + x^2)\,dz$*;*

(b) $\omega = (x + y^2 + z^3)\,dx + (y + z^2 + x^3)\,dy + (z + x^2 + y^3)\,dz$*.*

If we are in working with 2-forms, we have the same thing. If

$$\omega = P\,dx \wedge dy + Q\,dy \wedge dz + R\,dz \wedge dx,$$

then we have

$$d\omega = dP \wedge dx \wedge dy + dQ \wedge dy \wedge dz + dR \wedge dz \wedge dx. \tag{4.23}$$

Example 4.7. Let

$$\omega = (x^2 + z)\,dx \wedge dy = f\,dx \wedge dy,$$

where $f = x^2 + z$. Then

$$\begin{aligned} d\omega &= df \wedge dx \wedge dy = \\ &= (2x\,dx + \,dz) \wedge dx \wedge dy = 2x\,dx \wedge dx \wedge dy + \,dz \wedge dx \wedge dy = \\ &= dx \wedge dy \wedge dz. \end{aligned}$$

Example 4.8. If

$$\omega = (x + z^2)\,dx \wedge dy + (xyz)\,dy \wedge dz + (\cos y)\,dz \wedge dx,$$

then

$$\begin{aligned}
d\omega = d(x+z^2) \wedge\, dx \wedge\, dy + d(xyz) \wedge\, dy \wedge\, dz + d(\cos y) \wedge\, dz \wedge\, dx = \\
&= (2\,dx + 2z\,dz) \wedge\, dx \wedge\, dy + \\
&\qquad +(yz\,dx + xz\,dy + xy\,dz) \wedge\, dy \wedge\, dz + \\
&\qquad +(-\sin y\,dy) \wedge\, dz \wedge\, dx = \\
&= 2\,dx \wedge\, dx \wedge\, dy + 2z\,dz \wedge\, dx \wedge\, dy + \\
&\qquad +yz\,dx \wedge\, dy \wedge\, dz + xz\,dy \wedge\, dy \wedge\, dz + xy\,dz \wedge\, dy \wedge\, dz + \\
&\qquad -\sin y\,dy \wedge\, dz \wedge\, dx = \\
&= 2z\,dz \wedge\, dx \wedge\, dy + yz\,dx \wedge\, dy \wedge\, dz - \sin y\,dy \wedge\, dz \wedge\, dx = \\
&= 2z\,dx \wedge\, dy \wedge\, dz + yz\,dx \wedge\, dy \wedge\, dz - \sin y\,dx \wedge\, dy \wedge\, dz = \\
&= (2z + yz - \sin y)\,dx \wedge\, dy \wedge\, dz.
\end{aligned}$$

Exercise 13. *Let*

$$\omega = 3dx \wedge dy + (x^2 y)dy \wedge dz + \sin y dz \wedge dx.$$

Compute $d\omega$. You should find $d\omega = (2xy + \cos y)dx \wedge dy \wedge dz$.

Exercise 14. *If*

$$\omega = (\sin(xyz))\,dx \wedge\, dy + (x/z)\,dy \wedge\, dz + (e^{xy})\,dz \wedge\, dx,$$

compute $d\omega$.

As we saw in the previous examples, that are some terms that can be cancelled; this always happens. We have the following ***general formulas of differentiation***, that can be proved using the properties of the wedge product. If $f = f(x, y, z)$ is a function, then

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz. \tag{4.24}$$

If $\omega = P\,dx + Q\,dy + R\,dz$ is a 1-form, then

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy. \tag{4.25}$$

If $\omega = P\,dy \wedge\, dz + Q\,dz \wedge\, dx + R\,dx \wedge\, dy$ is a 2-form,

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\,dx \wedge\, dy \wedge\, dz. \tag{4.26}$$

We point out that the differential of a 0-form is a 1-form, the differential of a 1-form is a 2-form and the differential of a 2-form is a 3-form.

Exercise 15. *Do again the exercises 11(b), 12(a) and 13 using the equations* (4.24), (4.25) *and* (4.26).

If $f(x, y, z)$ is a function, we have

$$d^2 f = 0. \tag{4.27}$$

Similarly, if $\omega = f dx + g dy + h dz$, then

$$d^2 \omega = 0. \tag{4.28}$$

We sometimes use the compact notation $d^2 = 0$ for these two properties. In the next example we show both relations, and how both are an immediate consequence of the fact that, in partial derivatives of second order, the order of differentiation does not matter (for example $\partial^2 f/\partial x \partial y = \partial^2/\partial y \partial x$), as long as the double derivates are continuous.

Example 4.9. We will prove the two above relations. First, if $f = f(x, y, z)$, then

$$\begin{aligned} df &= \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz = \\ &= P\, dx + Q\, dy + R\, dz. \end{aligned}$$

When doing $d^2 f = d(df)$, the first term is

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z},$$

with $R = \frac{\partial f}{\partial z}$ and $Q = \frac{\partial f}{\partial y}$. Therefore we have

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} = 0.$$

The same is true for all the terms, and we obtain $d^2 f = 0$.

Now, let $\omega = P\, dx + Q\, dy + R\, dz$; we have

$$\begin{aligned} d\omega &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy \\ &= l\, dy \wedge\, dz + m\, dz \wedge\, dx + n\, dx \wedge\, dy. \end{aligned}$$

Then

$$d^2\omega = d(d\omega) = \left(\frac{\partial l}{\partial x} + \frac{\partial m}{\partial y} + \frac{\partial n}{\partial z}\right) dx \wedge\, dy \wedge\, dz.$$

Ignoring the $dx \wedge dy \wedge dz$, we see that we have

$$\begin{aligned} \frac{\partial l}{\partial x} + \frac{\partial m}{\partial y} + \frac{\partial n}{\partial z} &= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \\ &= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} = \\ &= 0, \end{aligned}$$

since $\partial^2 R/\partial x \partial y = \partial^2 R/\partial y \partial x$ and so on.

5 Differentation of vector fields

5.1 Differentiation

Here we will simply recall some of the way of differentiate functions and vector fields.

Let $\vec{\nabla}$ be the (formal) vector of the partial derivatives

$$\vec{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle = \frac{\partial}{\partial x}\,\vec{i} + \frac{\partial}{\partial y}\,\vec{j} + \frac{\partial}{\partial z}\,\vec{k}. \tag{5.29}$$

If f is a function, then we define the ***gradient*** of f as the vector field

$$\vec{\nabla} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle. \tag{5.30}$$

If $\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$ is a vector field, then we define the ***curl*** of $\vec{F}$ as the vector field

$$\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} = \tag{5.31}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}. \tag{5.32}$$

We define the ***divergence*** of $\vec{F}$ as the function

$$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \tag{5.33}$$

One can check that, if f is a function,

$$\operatorname{curl}(\vec{\nabla} f) = 0 \tag{5.34}$$

(technically, $\operatorname{curl}(\vec{\nabla} f) = \vec{0}$) and if $\vec{F}$ is a vector field

$$\operatorname{div}(\operatorname{curl} \vec{F}) = 0. \tag{5.35}$$

5.2 The correspondence with forms

The above costructions are apparently all different, but we can see now as they are all a translation of the differential for forms in the language of vector fields. First we need to see how we can associate forms to vector fields (and viceversa).

(a) We recall the ***musical isomorphism***. To a vector field $\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$ corresponds the form $Pdx + Qdy + Rdz$; to any form $\omega = Pdx + Qdy + Rdz$ corresponds the vector field $P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$.[1]

[1] Sometimes we use the notation $\vec{F}^\sharp$ for the form associated to the vector field $\vec{F}$ and $\omega^\flat$ for the vector field associated to the form ω. Observe that doing the both operations, one after the other, we get back to were we started, exactly as in music a flat and a sharp cancel each other.

We recall, equations (2.8) and (2.9), that we can introduce the formal vector field

$$d\vec{s} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

and that the 1-form corresponding to the vector field $\vec{F}$ is

$$\omega = \vec{F} \cdot d\vec{s}.$$

(b) If we consider the following 2-form $\omega = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$, we can associate the vector field $P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$.

One way of remembering this is that it is equivalent to applying the musical isomorphism and substituting the wedge product with a cross product (remember that the wedge product was modeled on the cross product). Thus, $dy \wedge dz$ corresponds to $\vec{j} \times \vec{k} = \vec{i}$, $dz \wedge dx$ corresponds to $\vec{k} \times \vec{i} = \vec{j}$, $dx \wedge dy$ corresponds to $\vec{i} \times \vec{j} = \vec{k}$. Conversely, to a vector field $P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$ we can associate the 2-form $P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$. Notice that, as already observed, $dy \wedge dz \neq dx$, so this is ***just a trick*** to help you remember.

Another way to think of this is to construct the formal vector field *element of area*

$$d\vec{A} = (dy \wedge dz)\,\vec{i} + (dz \wedge dx)\,\vec{j} + (dx \wedge dy)\,\vec{k}. \tag{5.36}$$

Then the 2-form corresponding to the vector field $\vec{F}$ is

$$\omega = \vec{F} \cdot d\vec{A}. \tag{5.37}$$

(c) To the function f we can associate the form $f\,dx \wedge dy \wedge dz$. As before, we can construct the *element of volume*

$$dV = dx \wedge dy \wedge dz, \tag{5.38}$$

and then

$$\omega = f\,dx \wedge dy \wedge dz = f\,dV. \tag{5.39}$$

If f is a function (or a 0-form), then $\vec{\nabla} f$ is the vector field corresponding to the 1-form df, using the correspondence in 1. (musical isomorphism). We can rewrite it as

$$df = (\vec{\nabla} f) \cdot d\vec{s}. \tag{5.40}$$

If to a vector field $\vec{F}$ we associate a form ω using the method in 1. (musical isomorphism), then $\operatorname{curl} \vec{F}$ corresponds to $d\omega$ (using the method in 2.). Again, this can be rewritten as

$$d(\vec{F} \cdot d\vec{s}) = (\operatorname{curl} \vec{F}) \cdot d\vec{A}. \tag{5.41}$$

Finally, if a vector field $\vec{F}$ and a form ω correspond by 2., then $\operatorname{div}\vec{F}$ corresponds to $d\omega$ using 3.. Again, this is

$$d(\vec{F}\cdot d\vec{A}) = (\operatorname{div}\vec{F})\,dV. \tag{5.42}$$

In this sense, the properties $d^2 = 0$ for forms correspond to $\operatorname{curl}(\vec{\nabla} f)) = 0$ and $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$ for vector fields.

6 Orientation and forms

The notion of orientation is the mathematical way of saying "how we move in a space", or "how we are in a space". As mentioned in section 1, we can think of dx as $\vec{i}$, that is a vector of magnitude 1 in the direction of the x-axis. In this sense, the dx corresponds to an orientation for the x-axis (the usual one). You can think of being on the x-axis, looking or walking in the direction of the numbers increasing. We can also go along the x-axis with the numbers decreasing, that is, in the $-\vec{i}$ or $-dx$ direction. In this sense, this corresponds to the orientation of the x-axis given by the 1-form $-dx$. This is the basic idea of the big correspondence between forms and orientations. We will see in this section how forms can be used to keep track of orientation also in two or three dimensions (or even more). This is the second reason why forms are a more natural object than vector fields.

We introduce the symbol $\wedge$, called ***wedge***, in order to keep track of orientation. Let us say that we are in the xy-plane, and that we are standing at the origin, looking in the direction $\vec{i}$ of the x-axis. Then the question is: where is the positive part of the y-axis? At our left or at our right? If we think the xy-plane in the space, in the first case we are standing with our body in the direction of the z-axis, in the second one we are upside down. These are the only two possible ways of being in the xy-plane. You can think that in the first case we are above, and in the second one we are doing spider-man walking upside down on the xy-plane. In the first case we say that the orientation is associated with $dx \wedge dy$, and in the second case with $dy \wedge dx$. You can also remember it this way. If you are walking counterclockwise in the xy-plane, in the first case the positive part of the x-axis comes first, and after 45° there is the positive part of the y-axis (x first, then y). In the second case, you have the positive part of the y-axis, and after 45° the positive part of the x-axis (y first, then x). In the same way we define $dy \wedge dz$, $dz \wedge dy$, $dx \wedge dz$ and $dz \wedge dx$.

Now, let us assume that we are in the space $\mathbb{R}^3$, oriented following the right hand rule. That is, we are standing at the origin, looking in the usual direction $\vec{i}$ of the x-axis, with the positive part of the y-axis at our left and with the head in the direction $\vec{k}$ of the z-axis. Then we say that this corresponds to $dx \wedge dy \wedge dz$.

Using this interpretation in terms of orientation, we can deduce again some of the properties that we have discussed before. For example, we have said that

$dx \wedge dy$ corresponds to us looking in the direction of $\vec{i}$ with $\vec{j}$ going out at our left. In this case, if we orient the y-axis with $-\vec{j}$ (we consider the number in decreasing order), then the vector $-\vec{j}$ is coming out at our right. In this case, the y-axis is oriented with $-dy$ and therefore this orientation corresponds to $(-dy) \wedge dx$. Indeed, there is first the negative part of the y-axis and then the positive part of the x-axis. But this is the natural orientation corresponding to $dx \wedge dy$. Therefore we have

$$(-dy) \wedge dx = dx \wedge dy.$$

Exercise 16. *With a reasoning similar to the one above, convince yourself that*

$$dy \wedge (-dx) = dx \wedge dy.$$

On the other hand, the only difference between the two orientations $dx \wedge dy$ and $dy \wedge dx$ is that one is the opposite of the other (we are upside down). So we have

$$dy \wedge dx = -dx \wedge dy.$$

Collecting the equalities we have

$$dx \wedge dy = (-dy) \wedge dx = dy \wedge (-dx) = -dy \wedge dx.$$

The same is true if we consider dz in the place of dx or dy.

We also have that

$$du \wedge du = 0,$$

when u is any variable. We can see that this second property agrees with the first one because, if $u = v$, then $du \wedge du = -du \wedge du$ (switching the two du's), which is possible only if $du \wedge du = 0$. This corresponds to the following idea: "to give an orientation to a plane you need two informations", you need to know the mutual position of x-axis and y-axis. With only the x-axis or only the y-axis, you do not know how you are standing. Or, if you prefer, if you consider $dx \wedge dx$, it means that at the left of the x-axis there is the x-axis again, which is impossible. Thus $dx \wedge dx = 0$.

7 Forms and integration

7.1 Integration of 2- and 3-forms

In this section we will say few words about forms and integration. We already know how to integrate 1-forms. To integrate 2-forms (or 3-forms) you ***check the orientation and ignore the wedge***. We will see how it works in two dimensions.

Example 7.1. Let us say that we have the region $D = [1,2] \times [3,4] = \{1 \leq x \leq 2,\ 3 \leq y \leq 4\}$ and that we want to integrate $x\,dx \wedge dy$. Then

$$\iint_D x\,dx \wedge dy = \iint_D x\,dx\,dy = \int_3^4 \int_1^2 x\,dx\,dy = \int_3^4 \frac{3}{4}\,dy = \frac{3}{4}.$$

We still wrote D in the integral when we removed the wedge because D is in the xy-plane which has the usual orientation, first x then y, which is the same of $dx \wedge dy$.

The natural question is: how is it possible that we can switch the order of integration in double integrals, changing for example from $dx\,dy$ to $dy\,dx$ without changing sign, but $dx \wedge dy = -\,dy \wedge dx$? Where did that negative sign go? The answer is: it is in the "check the orientation" part. We will illustrate this in an example.

Example 7.2. Let us consider the previous example, and let us change $dx \wedge dy$ with $-\,dy \wedge dx$.

$$\iint_D x\,dx \wedge\,dy = \iint_D x(-\,dy \wedge\,dx) = -\iint_D x\,dy \wedge\,dx$$

If we have $dy \wedge dx$, we are assuming that in our plane the y comes before the x. Then the region $D = [1,2] \times [3,4]$ must be written inverting x and y, so $D = [1,2] \times [3,4] = -[3,4] \times [1,2]$, and the integral changes a sign for compensating. The change of sign is because now we are upside down; if before we had to look down to see the rectangle, now we have to look up (think of being standing on the plane). You can also think of it this way. The region D comes with a natural upward orientation (first x than y it is upward by the right hand rule). If we are now orienting the plane with y preceeding x (with $dy \wedge dx$), our region D is oriented in the opposite way (the plane is oriented downwards). Therefore, when we split the integral in two line integrals, we are integrating on $-D$ (oriented downwards) and we need to add a negative sign. Now that we wrote $-D$, the orientation of the region is compatible with $dy \wedge dx$ and we can remove the wedge. Therefore,

$$\iint_D x\,dx \wedge\,dy = -\iint_D x\,dy \wedge\,dx = -\left(-\iint_{-D} x\,dy \wedge\,dx\right) =$$
$$= \int_1^2 \int_3^4 x\,dy\,dx = \int_1^2 x\,dx = \frac{3}{4}.$$

The one illustrated above is the actual process that happens every time you change the order of integration.

7.2 Change of variables and Jacobian

Using forms for integrating make more clear the process of the change of variables. For simplicity we will omitt the region where we are integrating.

In one variable, in the integral

$$\int P(x)dx$$

we do a change of variable (a u substitution) and write $x = f(u)$. Then we can compute the differential of x, and we find

$$dx = df = f'(u)du.$$

The integral becomes

$$\int P(f(u))f'(u)du.$$

The part $f'(u)$ comes precisely from the differential.

Before discussing the general case in two variables, we will do an example.

Example 7.3. Say that we have the double integral

$$\iint P(x, y)\, dx\, dy$$

and we want to do use polar coordinates. We know that the integral corresponds to

$$\iint P(x, y)\, dx \wedge\, dy$$

(because in the xy-plane the x comes before the y), so the question becomes what happens to the $dx \wedge dy$. We have that

$$\begin{cases} x = \rho \cos\theta, \\ y = \rho \sin\theta. \end{cases}$$

Therefore

$$\begin{cases} dx = d(\rho\cos\theta) = \cos\theta\, d\rho + \rho(-\sin\theta)\, d\theta = \cos\theta\, d\rho - \rho\sin\theta\, d\theta, \\ dy = d(\rho\sin\theta) = \sin\theta\, d\rho + \rho\cos\theta\, d\theta; \end{cases}$$

then

$$\begin{aligned} dx \wedge\, dy &= (\cos\theta\, d\rho - \rho\sin\theta\, d\theta) \wedge (\sin\theta\, d\rho + \rho\cos\theta\, d\theta) = \\ &= \cos\theta\sin\theta\, d\rho \wedge\, d\rho + \cos\theta\rho\cos\theta\, d\rho \wedge\, d\theta + \\ &\quad -\rho\sin\theta\sin\theta\, d\theta \wedge\, d\rho - \rho\sin\theta\rho\cos\theta\, d\theta \wedge\, d\theta = \\ &= \rho\cos^2\theta\, d\rho \wedge\, d\theta - \rho\sin^2\theta\, d\theta \wedge\, d\rho = \\ &= \rho\cos^2\theta\, d\rho \wedge\, d\theta + \rho\sin^2\theta\, d\rho \wedge\, d\theta = \\ &= \rho(\cos^2\theta + \sin^2\theta)\, d\rho \wedge\, d\theta = \\ &= \rho\, d\rho \wedge\, d\theta. \end{aligned}$$

Here you see why there is the ρ when you change to polar coordinates (the normal orientation for polar coodinates is to put ρ first, and then θ). So we find

$$\iint f(\rho\cos\theta, \rho\sin\theta)\rho\, d\rho\, d\theta.$$

In general, let us say that we have the integral

$$\iint f(x,y)\,dx\,dy,$$

corresponding to

$$\iint f(x,y)\,dx\wedge\,dy.$$

If we do the substitution

$$\begin{cases} x = X(u,v), \\ y = Y(u,v) \end{cases}$$

(with u coming first, then v), we get

$$\begin{cases} dx = \frac{\partial X}{\partial u}\,du + \frac{\partial X}{\partial v}\,dv, \\ dy = \frac{\partial Y}{\partial u}\,du + \frac{\partial Y}{\partial v}\,dv. \end{cases}$$

Then

$$\begin{aligned} dx\wedge\,dy &= \left(\frac{\partial X}{\partial u}\,du + \frac{\partial X}{\partial v}\,dv\right)\wedge\left(\frac{\partial Y}{\partial u}\,du + \frac{\partial Y}{\partial v}\,dv\right) = \\ &= \frac{\partial X}{\partial u}\frac{\partial Y}{\partial u}\,du\wedge\,du + \frac{\partial X}{\partial u}\frac{\partial Y}{\partial v}\,du\wedge\,dv + \\ &\quad + \frac{\partial X}{\partial v}\frac{\partial Y}{\partial u}\,dv\wedge\,du + \frac{\partial X}{\partial v}\frac{\partial Y}{\partial v}\,dv\wedge\,dv = \\ &= \left(\frac{\partial X}{\partial u}\frac{\partial Y}{\partial v} - \frac{\partial X}{\partial v}\frac{\partial Y}{\partial u}\right)\,du\wedge\,dv. \end{aligned}$$

On the other hand,

$$\frac{\partial X}{\partial u}\frac{\partial Y}{\partial v} - \frac{\partial X}{\partial v}\frac{\partial Y}{\partial u} = \det\begin{pmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{pmatrix}.$$

In the end we have

$$dx\wedge\,dy = \left(\frac{\partial X}{\partial u}\frac{\partial Y}{\partial v} - \frac{\partial X}{\partial v}\frac{\partial Y}{\partial u}\right)\,du\wedge\,dv = \det\begin{pmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{pmatrix}. \tag{7.43}$$

This is the reason why in the change of variables we have the determinant of the Jacobian. We have

$$\iint f(x,y)\,dx\,dy = \iint f(X(u,v),Y(u,v))\det\begin{pmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{pmatrix}\,du\,dv. \tag{7.44}$$

With similar computations (but longer), if in a triple integral in $dx\,dy\,dz$ we do a change of variables

$$\begin{cases} x = X(u,v,w), \\ y = Y(u,v,w), \\ z = Z(u,v,w), \end{cases}$$

we obtain

$$dx \wedge dy \wedge dz = \det \begin{pmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} & \frac{\partial X}{\partial w} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} & \frac{\partial Y}{\partial w} \\ \frac{\partial Z}{\partial u} & \frac{\partial Z}{\partial v} & \frac{\partial Z}{\partial w} \end{pmatrix} du \wedge dv \wedge dw, \tag{7.45}$$

and

$$\iiint f(x,y,z)\, dx\, dy\, dz =$$

$$= \iiint f(X(u,v,w), Y(u,v,w), Z(u,v,w)) \det \begin{pmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} & \frac{\partial X}{\partial w} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} & \frac{\partial Y}{\partial w} \\ \frac{\partial Z}{\partial u} & \frac{\partial Z}{\partial v} & \frac{\partial Z}{\partial w} \end{pmatrix} du\, dv\, dw. \tag{7.46}$$

7.3 Stokes and Gauss

The last thing we will mention in these notes is the interpretation of Stokes' theorem and Gauss' theorem in terms of forms.

Let $f(x)$ be a function on an interval $[a,b]$, and let $f'(x)$ be its derivative. The funtamental theorem of calculus says

$$\int_{[a,b]} f'(x)\, dx = f(b) - f(a).$$

We saw that $f'(x)\, dx$ is the 1-form $f'(x)\, dx = df$. Also, the orientation dx on that interval gives a vector coming out of b and coming in a. Therefore, the orientation induced on the two end points is positive on b and negative on a. So we can rewrite the fundamental theorem of calculus as

$$\int_{[a,b]} df = f(b) - f(a). \tag{7.47}$$

Let $\vec{F} = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$ be a vector field on a surface D in the space, bounded by a (simple) closed curve C with the induced orientation. Then, Stokes' theorem says

$$\iint_D \operatorname{curl} \vec{F} \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{s}.$$

We saw in the equation (5.41) that, if ω is the 1-form $\omega = \int_C \vec{F} \cdot d\vec{s}$, then $\operatorname{curl} \vec{F} \cdot d\vec{A} = d\omega$. So, we can rewrite the above theorem as

$$\iint_D d\omega = \int_C \omega. \tag{7.48}$$

Finally, if $\vec{F} = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$ is a vector field on a solid E in the space, bounded by a (simple) closed surface S with the induced orientation, Gauss' theorem says

$$\iiint_E (\operatorname{div}\vec{F})\,dV = \iint_S \vec{F}\cdot d\vec{A}.$$

Again, if $\omega = \vec{F}\cdot d\vec{A}$, by (5.42), $\operatorname{div}\vec{F}\,dV = d\omega$, and we can rewrite Gauss' theorem in terms of forms as

$$\iiint_E d\omega = \iint_S \omega. \tag{7.49}$$

Therefore, all these theorems are actually the same (in different dimensions) that simply says "the integral cancels d". This is called the ***generalized Stokes' theorem***.