Geometry vs Calculus

In these notes we are only going to work in the two dimensional case, but a smilar treatment is possible in any dimension.

1 Geometry

1.1 Curves

We call a ***curve*** a 1-dimensional (nice) subset of the plane, which can be parametrized by some function $\vec{r}(t) = (x(t), y(t))$, with $a \leq t \leq b$. The points $\vec{r}(a) = A$ and $\vec{r}(b) = B$ are the end-points of the curve. We say that the curve is ***closed*** if $\vec{r}(a) = \vec{r}(b)$, and ***open*** otherwise.

We say that a curve is ***simple*** if it does not self intersect.

1.2 Regions

A region D in the plane is ***open*** if, for any point (x, y) in D, we can put a small ball around (x, y) interely contained in D. Usually, a region will be open if it is described by strict inequalities. We will always work with open regions.

An open region D is ***connected***, if any two points can be connected by a curve (called a path). This means that D is only "one piece". We point out that this is the definition of connected for *open* regions, not for generic regions.

An open region D is ***simply connected*** if it is connected and it has no holes.

2 Vector fields

2.1 Definitions

A vector field $\vec{F}$ is ***exact***, if there is a function f such that

$$\vec{F} = \vec{\nabla} f.$$

A vector field $\vec{F}$ is ***conservative*** if the integral of $\vec{F}$ along any curve C only depends on the endpoint of C.

A vector field $\vec{F} = \langle P, Q \rangle$ is ***closed*** if

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

2.2 Properties

Proposition 2.1. An exact vector field is conservative.

Proof. By the fundamental theorem of calculus for line integrals. Let C be a curve from A to B, with parametrization $\vec{r}(t)$, $a \leq t \leq b$ and $\vec{r}(a) = A$, $\vec{r}(b) = B$. Let $\vec{F} = \vec{\nabla} f$ be an exact vector field. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(B) - f(A),$$

which only depends on A and B. $\square$

Proposition 2.2. Let $\vec{F}$ be a vector field. The following are equivalent:

(a) $\vec{F}$ is conservative;

(b) the integral of $\vec{F}$ along any closed curve is zero;

(c) the integral of $\vec{F}$ along any simple closed curve is zero.

Proof. First notice that if the integral along any closed curve is zero, then in particular the integral any simple closed curve is zero. Conversely, each closed curve can be cut as an union of simple closed curves, therefore, if the integral on each part is zero, the overall integral is zero. This proves that the last two conditions are equivalent.

Second, let $\vec{F}$ be conservative, and we will prove that the integral along any closed curve is zero. Let C be a closed curve, and choose any two distinct points A and B on C. These two points cut C in two curves C_1 and $-C_2$, say, the first one from A to B and the second one from B to A. We have that C is the sum of C_1 and $-C_2$. Therefore

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} = \int_{C_1+(-C_2)} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = \\ = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}.\end{aligned}$$

On the other hand, since $\vec{F}$ is conservative and C_1 and C_2 both go from A to B,

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}.$$

Hence

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0.$$

The remaing implication works exactly as the above one. We assume that the integral of $\vec{F}$ along any closed curve is zero, and we will prove that the integral only depend on the endpoints. Let C_1, C_2 be two curves with the same end-points, both from A to B. If we glue them together, but changing the direction of C_2, that is, going from B to A, we obtain a closed curve C. Now C is the sum of C_1 and $-C_2$. Since the integral of $\vec{F}$ along C is zero,

$$\begin{aligned} 0 &= \int_C \vec{F}\cdot d\vec{r} = \int_{C_1+(-C_2)} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{-C_2} \vec{F}\cdot d\vec{r} = \\ &= \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r}, \end{aligned}$$

that is,

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}.$$

□

Proposition 2.3. An exact vector field is closed.

Proof. If $\vec{\nabla} f = \vec{F} = \langle P, Q\rangle$, then $P = \partial f/\partial x$ and $Q = \partial f/\partial y$. Then

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial y}.$$

□

3 Geometry vs Calculus

The next property is naturally the first one of this section, because is the one with the weakest geometric assumptions, but is the one with the hardest proof.

Proposition 3.1. Let $\vec{F}$ be a vector field defined on an ***open*** region D. If $\vec{F}$ is conservative, then it's closed.

Proof. Suppose that $\vec{F}$ is not closed. We will see how this leads to a contradiction. If $\vec{F}$ is not closed, there is a point (x_0, y_0) where

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \neq 0.$$

We can assume that that value is actually positive. If it is negative, then proof proceed in the same way. So let

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} > 0$$

at (x_0, y_0). By continuity and by the ***openess*** of D, there is a ball B in D where $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} > 0$. Let C be the boundary of B (positively oriented). Then C is a simple closed curve contained in D. Since $\vec{F}$ is conservative,

$$\int_C \vec{F}\cdot d\vec{r} = 0.$$

If we combine the above equality with Green's theorem (and with what we said about B),

$$0 = \int_C \vec{F} \cdot d\vec{r} = \iint_B \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy > 0,$$

and this is a contradiction. $\square$

Proposition 3.2. Let $\vec{F}$ be a vector field defined on an ***open connected*** region D. If $\vec{F}$ is conservative, then it's exact.

Proof. We can fix any point (x_0, y_0) in D. For any point (x, y) in D, there is a path C that connects (x_0, y_0) with (x, y), since D is ***open connected***. Since $\vec{F}$ is conservative, the integral

$$\int_C \vec{F} \cdot d\vec{r}$$

only depends on the end-points (x_0, y_0) and (x, y). Therefore, we can define the following function

$$f(y, x) = \int_{(x_0,y_0)}^{(x,y)} \vec{F} \cdot d\vec{r},$$

where the integral is taken on any curve from (x_0, y_0) to (x, y). We emphasize that the function $f(x, y)$ is well-defined since $\vec{F}$ is conservative, and can be defined for any point since D is open connected.

It is possible to check (it is on the book, a little technical) that

$$\vec{\nabla} f = \vec{F}.$$

$\square$

Essentially, we wanted the integral to be a solution, as in the one-dimensional case, and the assumptions on D and $\vec{F}$ guarantee that we can do it.

Proposition 3.3 (Poincare's theorem)**.** Let $\vec{F}$ be a vector field defined on an ***open simply connected (hence connected)*** region D. If $\vec{F}$ is closed, then it's exact.

Proof. Since D is ***open connected***, it is enough to prove that $\vec{F}$ is conservative. Then, by the previous result, it will also be exact.

Let $\vec{F} = \langle P, Q \rangle$ be closed, that is, with

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

Let C be any simple closed curve in D. Since D is ***simply connected***, C is the boundary of a region B contained in D. We can assume that C is positively oriented, otherwise there is only a difference in sign. By Green's theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint_B \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0.$$

$\square$

4 Usage in "real" life

How do we use the previous results? If we always assume our vector fields to be defined over an ***open*** region D, we have the following implications

$$\begin{array}{c} \text{exact} \\ {\scriptstyle D \text{ connected}} \Uparrow \Downarrow \\ \text{conservative} \\ {\scriptstyle D \text{ simply connected}} \Uparrow \Downarrow \\ \text{closed.} \end{array}$$

If you want to check if a vector field $\vec{F}$ is exact on a region D the general strategy is the following. First, if $\vec{F}$ closed? If no, than it cannot be exact. If it is closed, look at the region. If D is open simply connected, then $\vec{F}$ is exact and you are done. If D is not open simply connected, then you have to try to find an f such that $\vec{\nabla} f = \vec{F}$.

Usually, it does not pay off to try to check if the vector field is conservative. It is almost impossible to show directly that the integral is 0 along *any* closed curve C. Even if you suspect your vector field not conservative (hence not exact), it may not be clear how to find a closed curve where the integral is non zero (and anyway, you need to compute the integral).

5 Examples

5.1 An exact vector field

By definition, for any function f, $\vec{\nabla} f$ is exact. For example, if $f = xy$,

$$\vec{\nabla} f =< y, x >$$

is exact.

5.2 A vector field that is not closed (hence nor conservative, nor exact)

Let

$$\vec{F} =< -y, x > .$$

If we compute the mixed derivatives, $\partial Q/\partial x = 1$, while $\partial P/\partial y = -1$.

5.3 A closed vector field not conservative (hence nor exact)

This example is on the book. Let

$$\vec{F} = \frac{1}{x^2 + y^2} < -y, x > .$$

Then

$$\frac{\partial Q}{\partial x} = \frac{y^2 - x^2}{x^2 + y^2} = \frac{\partial P}{\partial y}.$$

This means that $\vec{F}$ is closed. However, if C is the unit circle (considered counterclockwise)

$$\int_C \vec{F} \cdot d\vec{r} = 2\pi.$$

Therefore, $\vec{F}$ is not conservative. Notice that the domain of $\vec{F}$ is $\mathbb{R}^2 - \{(0,0)\}$ which is not simply connected.

5.4 A conservative vector field that is not exact

Let

$$\vec{F} = < \frac{1}{x}, y > .$$

Notice that the domain of $\vec{F}$ is everything except the y-axis (where $x = 0$). Thus, the domain of $\vec{F}$ is not connected, but constists of two pieces. Each piece, however, is simply connected.

On the first piece $\{x > 0\}$, $\vec{F}$ is exact, and we can explicitely find an antiderivative (a potential). If

$$f_1(x,y) = \ln(x) + y^2/2,$$

then $\vec{\nabla} f_1 = \vec{F}$. Notice that f_1 is only defined for $x > 0$. Similarly on the second piece $\{x < 0\}$, we have

$$f_2(x,y) = -\ln(-x) + y^2/2,$$

with $\vec{\nabla} f_2 = \vec{F}$. Again, f_2 is only defined for $x < 0$.

Each closed curve C in the domain of $\vec{F}$ has to be contained in one of the two pieces $\{x > 0\}$ or $\{x < 0\}$, where $\vec{F}$ has a potential. Therefore $\int_C \vec{F} \cdot d\vec{r} = 0$. However, there is no way of gluing f_1 and f_2 (or $f_2 + c$ for some constant c) in a way that is defined on all the plane. Therefore $\vec{F}$ is conservative but not exact.