Midterm review

1. Let $\sigma = (1\,2)$ and $\tau = (3\,4\,5)$ in S_5, and let $H = \langle \sigma, \tau \rangle$. Show that $H \cong \mathbb{Z}/2 \times \mathbb{Z}/3$. Deduce from this that H is cyclic.

2. Let $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{R})$. Show that $a^4 = 1$ and $b^3 = 1$ but ab has infinite order, hence $\langle a, b \rangle$ is an infinite group.

3. Show that an infinite group has a proper non-trivial subgroup.

4. Let $\varphi : G \to H$ be a group homomorphism; let $K = \ker \varphi \leq G$. For each $g \in G$, recall that we have the conjugation isomorphism $i_g : G \to G$, $i_g(x) = gxg^{-1}$. Show that, for every $g \in G$, $i_g(K) = K$.